A-infinity GL(N)-equivariant matrix integrals-III. Tree-level calculations and variations of nc-Hodge structure on complex projective manifolds.

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Tree level BV on complex projective manifold (g=0 calculations)

- nc-BV on $Symm(C_\lambda[1+d])$, $C_\lambda = \oplus_{j=0}^{\infty}(U[1]^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}}$

$$\hbar\Delta S + \frac{1}{2}\{S,S\} = 0,\ S = \sum_{g\geq 0,i} \hbar^{2g-1+i} S_{g,i}\,,\quad S_{g,i} \in Symm^i(C_\lambda[1+d]),$$

$$\Leftrightarrow \Delta \exp(S/\hbar) = 0$$

$$\{S_{0,1}, S_{0,1}\} = 0, \text{- } A_\infty - \text{CY algebra}$$

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$$u\Delta\gamma + \bar{\partial}\gamma + \frac{1}{2}[\gamma, \gamma] = 0,$$

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- The component

$$\gamma_0^{cl} \in \Omega^{0,1}(M, T)$$

  $\bar{\partial}\gamma_0^{cl} + \frac{1}{2}[\gamma_0^{cl}, \gamma_0^{cl}] = 0$- deformations of complex structure on M

Noncommutative periods map (B[5])

- $$u\Delta\gamma + \bar{\partial}\gamma + \frac{1}{2}[\gamma, \gamma] = 0, \Leftrightarrow (u\Delta + \bar{\partial})\exp(\frac{1}{u}\gamma) = 0$$

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-local system of $(u\Delta + \bar{\partial})-$cohomology over $\mathbb{A}^1_u \backslash \{0\}$
- $$\begin{aligned} \Omega(t, u) &= \int_M r_u \exp(\frac{1}{u}\gamma) \vdash \varpi_0 \\ \Omega(t, u) &\in H^*_{DR}(M)((u)) \widehat{\otimes} \mathcal{O}_{\mathcal{M}_{\Lambda T}} \end{aligned}$$
$t \in \mathcal{M}_{\Lambda T}$, $T_0\mathcal{M}_{\Lambda T} = H^*(M, \Lambda^* T)$

Noncommutative periods map and g=0 Gromov-Witten of the mirror

- $$\Omega(t,u) = \int_M r_u \exp(\frac{1}{u}\gamma) \vdash \varpi_0$$

  For γ^W, normalized using a filtration W opposite to F^{Hodge}([B5],1999):

  $$\frac{\partial^2}{\partial t^i \partial t^j}\Omega^W = u^{-1} C_{ij}^k(t) \frac{\partial}{\partial t^k}\Omega^W$$

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- $$C_{kij}(t) = \partial^3_{kij}(\text{genus} = 0 \quad \text{GW-potential of} \quad M^{mirror})$$

Semi-infinite (Noncommutative) Hodge structures ([B5]).

- The class $[\exp(\gamma^{W})\varpi] = \Omega(t,u)$ is obtained as intersection of moving subspace

$$\Omega(t,u) = \mathcal{L}(t) \cap (\text{Affine space}(W))$$
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- Over mod space of complex structures it reduces to VHS

Noncommutative Hodge structures ([B5]), cont'd

- $\mathcal{L}(t)$ corresponds via HKR and formality isomorphisms for $C^*(A, A) + k[\xi, \frac{\partial}{\partial \xi}]-$module $C_*(A)$ to

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- Real structure on HP in the case of arbitrary $A_\infty-$algebra?

Summation over trees, BCOV lagrangian and BV cyclic operad.

- For $W = \overline{F^{Hodge}}$, $\gamma(t) = \Sigma\gamma_a t^a + \Delta\alpha(t)$, $\Delta(\gamma) = 0$

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- Meaning of this : the tree-level Feynman transform of the operad of BV algebras / $\Delta = 0$= operad of $H_*(\mathcal{M}_{0,n})$

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